spectral sequence

Let J be the category $\mathbb{Z} \cup \{-\infty, +\infty\}$ with order.

$Fl_n(J)$ is the category of n composable morphisms.

D a triangulated category, define a spectral object

is $(X: Fl_1 \to D,\ \delta: Fl_2 \to \text{triangles})$

such that $(\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot) \mapsto (X_f \to X_{g\circ f} \to X_g \to X_f[1])$

We can obtain spectral sequence according to these data.

Taking cohomology, we get a long exact sequence:

$$\cdots \to H^{n-1}(g) \to H^n(f) \to H^n(g\circ f) \to H^n(g) \to H^{n+1}(f) \to \cdots$$

So we get the boundary $B[{}^{\ n}_{f,g}] = \mathrm{im}(H^{n-1}(g) \to H^{n+1}(f))$

and the cycle $Z[{}^{\ n}_{f,g}] = \ker(H^n(g) \to H^{n+1}(f))$.

<u>Prop</u> If one has $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot \in Fl_3$, we have

$$B[{}^{\ n}_{g,h}] \subset Z[{}^{\ n}_{f,g}] \subset H^n(g).$$

Thus we can define $E[{}^{\ \ n}_{f,g,h}] = Z[{}^{\ n}_{f,g}] / B[{}^{\ n}_{g,h}]$.

(<u>Pf</u>. the diagram $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "id"'] & \cdot \arrow[r, "g"] \arrow[d, "id"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "f"] & \cdot \arrow[r, "hg"] & \cdot \end{tikzcd}$ gives $H^n(g) \to H^{n+1}(f)$ pass through $H^n(hg)$,

$\Rightarrow$ composition w/ $H^{n-1}(h) \to H^n(g)$ is zero.)

The definition of differential:

Take $\xrightarrow{f_2} \xrightarrow{f_3} \xrightarrow{f_4} \in Fl_3$. the diagram $\begin{tikzcd} \cdot \arrow[r, "f_2"] \arrow[d, "id"'] & \cdot \arrow[r, "f_3"] \arrow[d, "id"] & \cdot \arrow[d, "f_4"] \\ \cdot \arrow[r, "f_2"] & \cdot \arrow[r, "f_4 f_3"] & \cdot \end{tikzcd}$ gives

$$\begin{tikzcd} H^n(f_3) \arrow[r] \arrow[d] & H^{n+1}(f_2) \arrow[d, "id"] \\ H^n(f_4 f_3) \arrow[r] & H^{n+1}(f_2) \end{tikzcd}$$

, the diagram $\begin{tikzcd} \cdot \arrow[r, "f_2"] \arrow[d, "id"'] & \cdot \arrow[r, "f_4 f_3"] \arrow[d, "f_3"] & \cdot \arrow[d, "id"] \\ \cdot \arrow[r, "f_3 f_2"] & \cdot \arrow[r, "f_4"] & \cdot \end{tikzcd}$ gives

$$\begin{tikzcd} H^n(f_4 f_3 f_2) \arrow[r] \arrow[d, "id"] & H^n(f_4 f_3) \arrow[r] \arrow[d] & H^{n+1}(f_2) \arrow[d] \\ H^n(f_4 f_3 f_2) \arrow[r] & H^n(f_4) \arrow[r] & H^{n+1}(f_3 f_2) \end{tikzcd}$$

Here we obtain

where row & column are exact.

$$\begin{array}{ccccc} & & H^n(f_3) & \xrightarrow{id} & H^n(f_1) \\ & & \downarrow u & & \downarrow \delta \\ H^n(f_4f_3f_2) & \to & H^n(f_4f_3) & \to & H^{n+1}(f_2) \\ \downarrow id & & \downarrow & & \downarrow \\ H^n(f_4f_3f_2) & \to & H^n(f_4) & \to & H^{n+1}(f_3f_2) \end{array}$$

This gives the map

$$\psi : Z[f_3 \overset{n}{,} f_4] = \mathrm{im}(H^n(f_4f_3) \to H^n(f_4)) = H^n(f_4f_3)/\mathrm{im}\, u \to H^{n+1}(f_2)/\mathrm{im}\,\delta$$

$\ker\psi \simeq \ker(H^n(f_4) \to H^{n+1}(f_3f_2)) = Z[f_3f_2 \overset{n}{,} f_4]$,

$\mathrm{im}\,\psi = \mathrm{im}(H^n(f_4f_3) \to H^{n+1}(f_2))/\mathrm{im}\,\delta = B[f_2 \overset{n+1}{,} f_4f_3]/B[f_2 \overset{n+1}{,} f_3]$.

Thus we obtain the isomorphism $Z[f_3 \overset{n}{,} f_4]/Z[f_3f_2 \overset{n}{,} f_4] \xrightarrow{\sim} B[f_2 \overset{n+1}{,} f_4f_3]/B[f_2 \overset{n+1}{,} f_3]$.

If we have $\xrightarrow{f_1}\xrightarrow{f_2}\xrightarrow{f_3}\xrightarrow{f_4}\xrightarrow{f_5} \in Fl_5$, define the differential by

$$E[f_3, \overset{n}{f_4}, f_5] = Z[f_3 \overset{n}{,} f_4]/B[f_4 \overset{n}{,} f_5] \twoheadrightarrow Z[f_3 \overset{n}{,} f_4]/Z[f_3f_2 \overset{n}{,} f_4]$$

$$\xrightarrow{\sim} B[f_2 \overset{n+1}{,} f_4f_3]/B[f_2 \overset{n+1}{,} f_3] \hookrightarrow Z[f_1 \overset{n+1}{,} f_2]/B[f_2 \overset{n+1}{,} f_3] = E[f_1, \overset{n+1}{f_2}, f_3].$$

Prop For $\xrightarrow{f_1} \cdots \xrightarrow{f_7} \in Fl_7$, the composition of

$E[f_5, \overset{n-1}{f_6}, f_7] \xrightarrow{d} E[f_3, \overset{n}{f_4}, f_5] \xrightarrow{d} [f_1, \overset{n+1}{f_2}, f_3]$ is zero, and the cohomology is isom to $E[f_3f_2, \overset{n}{f_4}, f_6f_5]$.

(Pf. $\ker = Z[f_3f_2 \overset{n}{,} f_4]/B[f_4 \overset{n}{,} f_5]$, $\mathrm{im} = B[f_4 \overset{n}{,} f_6f_5]/B[f_4 \overset{n}{,} f_5]$,

thus $H = \ker/\mathrm{im} = Z[f_3f_2 \overset{n}{,} f_4]/B[f_4 \overset{n}{,} f_6f_5] = E[f_3f_2, \overset{n}{f_4}, f_6f_5]$.)

Two types of notation:

$E[p-r, \overset{n}{p-1}, p, p+r-1] = I_r^{-p,n+p}$

or $I_r^{p,q} = E[-p-r, \overset{p+q}{-p-1}, -p, -p+r-1]$

$d_r : I_r^{p,q} \to I_r^{p+r,q-r+1}$, $r \geq 1$

$E[p-r+1, \overset{n}{p}, p+1, p+r] = II_{r+1}^{n-p,p}$

or $II_r^{p,q} = E[q-r+2, \overset{p+q}{q}, q+1, q+r-1]$

$d_r : II_r^{p,q} \to II_r^{p+r,q-r+1}$, $r \geq 2$.

Convergence

Consider the objects $E[-\infty, p, \overset{n}{q}, +\infty]$.

They gives the filtration of objects $H^n(-\infty, +\infty)$ as follows:

let $F^p H^n(-\infty,+\infty) = \mathrm{im}(H^n(-\infty, p) \to H^n(-\infty,+\infty))$, then

$$E[-\infty, p, \overset{n}{q}, +\infty] \simeq F^q H^n(-\infty,+\infty) / F^p H^n(-\infty,+\infty).$$

Proof omitted. (Thus the associated graded is $E[-\infty, p, \overset{n}{p+1}, +\infty] = E_\infty^{p,q}$.)

When $E[r, p, \overset{n}{q}, s]$ are isomorphic when $-\infty \in r \leq v(n+1)$ and $u(n-1) \leq s \leq +\infty$, the spectral sequence is convergence, and write $E_r^{p,q} \Rightarrow H^{p+q}(-\infty, +\infty)$.

(This is true if $H^n(p,q) = 0$ for $u(n) \leq p \leq q \leq +\infty$ and
$H^n(p,q) = 0$ for $-\infty \leq p \leq q \leq v(n)$.)

Example
- For double complex, see [Verdier]
- Filtration $0 = X(-\infty) \subset \cdots \subset X(p) \subset X(p+1) \subset \cdots \subset X(+\infty) = X$,

  so we can get such a spectral object.
- apply of a cohomology functor

  For the filtration $Y(q) = \cdots \to 0 \to Y^{-q} \to Y^{-q+1} \to \cdots$,

  apply the cohomology functor $F^p : K(A) \to B$ to get $I_1^{p,q} = \boxed{F^q(Y^p)}$.

  apply to the filtration $X(q) = \cdots \to X^{q-3} \to X^{q-2} \to \ker(d) \to 0 \to \cdots$

  to get $II_2^{p,q} = \boxed{F^p(H^q(X))}$. note that they are different
- Composition of functors

  For three abelian categories $\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$, and $A \in \mathcal{C}$.

use the injective resolution $A \to C(A)$ and assume F map injectives to G-acyclic, we get two spectral sequence $I_2^{p,q} = R^p((R^q G) \circ F)(A)$, $II_2^{p,q} = R^p G(R^q F(A))$.

I degenerate and they converge to the same object.

Long exact sequence in the case $E_2^{p,q}=0$ if $p<0$ or $q<0$.

$$0\to E_2^{1,0}\to L^1\to E_2^{0,1}\xrightarrow{d}E_2^{2,0}\to\ker(L^2\to E_2^{0,2})\to E_2^{1,1}\xrightarrow{d}E_2^{3,0}$$

This can be written in the form $E[r,p,\overset{n}{q},s]$ to get a longer infinitely many LES. see [Verdier].

Hochschild - Serre spectral sequence

Recall group cohomology: $\mathrm{Rep}(G)=\mathrm{Coh}(BG)$. $H^i(G,A)=H^i(BG,A)$.

Let G be a group, H its closed normal subgroup.

we have the Cartesian diagram

$$\begin{tikzcd} BH \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BG \arrow[d, "f"] \\ pt \arrow[r] & B(G/H) \arrow[r, "g"] & pt \end{tikzcd}$$

Consider the composition functors $g_*\circ f_*$.

f_*A as G/H-module, is $H^\cdot(H,A)$ when A regarded as H-mod.

Similarly, For a G-torsor $f: X\to BG$, $\mathcal{F}\in\mathrm{Coh}(X)$,

Consider the composition $g_*\circ f_*$, by base change,

$f_*\mathcal{F}$ as G-module is $H^\cdot(X',\mathcal{F})$, where $X'=X\times_{BG}pt$.

<u>Def</u>. X a variety over k, $X^s=X\times_{\mathrm{Spec}\,k}\mathrm{Spec}\,k_s$.

algebraic part of Brauer group of X is

$$\mathrm{Br}_1(X)=\ker(\mathrm{Br}\,X\to\mathrm{Br}\,X^s)$$

<u>Prop</u> we have the long exact sequence

$$0\to\overset{H^1(G_k,k_s^\times)}{\overset{\parallel}{0}}\to\underset{H^1(X,\mathbb{G}_m)}{\underset{\parallel}{\mathrm{Pic}\,X}}\to\overset{H^0(G_k,H^1(X^s,\mathbb{G}_m))}{\overset{\parallel}{(\mathrm{Pic}\,X^s)^{G_k}}}\to\underset{H^2(G_k,H^0(X^s,\mathbb{G}_m))}{\underset{\parallel}{\mathrm{Br}_k}}\to\overset{\ker(H^2(X,\mathbb{G}_m)\to H^2(X^s,\mathbb{G}_m))}{\overset{\parallel}{\mathrm{Br}_1(X)}}\to\underset{H^1(G_k,H^1(X^s,\mathbb{G}_m))}{\underset{\parallel}{H^1(G_k,\mathrm{Pic}\,X^s)}}\to H^3(k,\mathbb{G}_m).$$

Extra: ① proof of the filtration

Lemma:
$$\begin{tikzcd} & Y \arrow[d, "\phi"] \arrow[dr, "\chi"] & \\ X' \arrow[ur] \arrow[r, "\phi'"'] & X \arrow[r, "\eta"'] & X'' \end{tikzcd}$$
row exact, then $\bar{\eta}: \operatorname{im}\phi / \operatorname{im}\phi' \xrightarrow{\sim} \operatorname{im}\chi$.

exact $\Rightarrow \operatorname{im}\phi' = \ker\eta$ $\quad \eta(\operatorname{im}\phi) = \operatorname{im}\chi$, $\quad \operatorname{im}\phi' \subset \operatorname{im}\phi$. $\square$

Apply it to the case
$$\begin{tikzcd} & H^n(f_2f_1) \arrow[d] \arrow[dr] & \\ H^{n-1}(f_3) \arrow[ur, "\delta"] \arrow[r, "\delta"] & H^n(f_2) \arrow[r] & H^n(f_3f_2) \end{tikzcd}$$

we get $E[f_1, f_2, \overset{n}{f_3}] = \operatorname{im}(H^n(f_2f_1) \to H^n(f_2)) / \operatorname{im}(H^{n-1}(f_3) \to H^n(f_2)) \simeq \operatorname{im}(H^n(f_2f_1) \to H^n(f_3f_2))$.

Apply to the case
$$\begin{tikzcd} & H^n(f_2f_1) \arrow[d] \arrow[dr] & \\ H^n(f_1) \arrow[ur] \arrow[r] & H^n(f_3f_2f_1) \arrow[r] & H^n(f_3f_2) \end{tikzcd}$$

we get $E[f_1, \overset{n}{f_2}, f_3] = \operatorname{im}(H^n(f_2f_1) \to H^n(f_3f_2)) \simeq \operatorname{im}(H^n(f_2f_1) \to H^n(f_3f_2f_1)) / \operatorname{im}(H^n(f_1) \to H^n(f_3f_2f_1))$.

In conclusion, $E[-\infty, \overset{n}{p, q}, +\infty] = \operatorname{im}(H^n(-\infty, q) \to H^n(-\infty, +\infty)) / \operatorname{im}(H^n(-\infty, p) \to H^n(-\infty, +\infty))$.

$$\begin{tikzcd} \bullet \arrow[r, "f_2f_1"] \arrow[d, "f_1"'] & \bullet \arrow[d, "id"] \\ \bullet \arrow[r, "f_2"] \arrow[d, "id"'] & \bullet \arrow[d, "f_3"] \\ \bullet \arrow[r, "f_3f_2"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "f_2f_1"] \arrow[d, "id"'] & \bullet \arrow[d, "f_3"] \\ \bullet \arrow[r, "f_3f_2f_1"] \arrow[d, "f_1"'] & \bullet \arrow[d, "id"] \\ \bullet \arrow[r, "f_3f_2"] & \bullet \end{tikzcd}$$

② Study of the examples.

$Y(q) = \quad 0 \to \cdots \to Y^{-q} \to \cdots \to$

$p < q$ $\quad X(p,q) = \cdots \to Y^{-q} \to \cdots \to Y^{-p-1} \to \cdots$

$$\begin{tikzcd} \cdots \arrow[r] & Y^{-q-1} \arrow[d] \\ & Y^{q} \arrow[r] & \cdots \end{tikzcd}$$

δ $\quad X(q, r) \to X(p,q)[1]$

apply $F^{\cdot}$: $I_1^{p,q} = F^{p+q} X(-p-1, -p) = F^{p+q}(Y^p[-p]) = F^q(Y^p)$.

$d_1 : I_1^{p,q} \to I_1^{p+1,q}$ is just $F^q(Y^p) \to F^q(Y^{p+1})$. $I_2^{p,q} = H^p(F^q(Y^{\cdot}))$

$Y^{\cdot} = FC(A)$, $F = R^{\cdot}G$. $\quad F^q(Y^{\cdot}) = (R^qG \circ F)(C(A))$. $I_2^{p,q} = R^p(R^qG \circ F)(A)$.

$Y(q) = \cdots \to Y^{q-2} \to \ker(d^{q-1}) \to 0 \to \cdots$

$X(p,q) = 0 \to \operatorname{im}(d^{p-1}) \to \cdots \to Y^{q-2} \to \ker(d^{q-1}) \to 0 \to \cdots$ $\quad X(p, p+1) = H^p(Y)[-p]$

apply coh. functor, then $II_2^{p,q} = F^{p+q}(H^q(Y)[-q]) = F^p(H^q(Y))$.